Strictly convex norms and topology

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Strictly convex norms

Definition (Clarkson, 1936)

A norm $\|\cdot\|$ is called *strictly convex* if $x = y$ whenever $\|x\| = \|y\| = \frac{1}{2}\|x+y\|$.

Natural norms are usually not strictly convex, but often a space can be given an *equivalent* strictly convex norm.

We concentrate on strictly convex *dual* norms on dual spaces.

Theorem (Šmulyan, 1940)

If $\|\cdot\|$ is a strictly convex *dual norm* on X^* then the predual $\|\cdot\|$ on X is Gâteaux smooth.

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Characterizing strict convexity

Question

Given a (dual) Banach space, does it admit a strictly convex (dual) norm?

Yes if e.g. X is separable, reflexive (more generally weakly compactly generated) or if $X = L_1(\mu)$.

No if e.g. $X = \ell_\infty^c(\Gamma)$, Γ uncountable.

Figures such as Day, Lindenstrauss, Mercourakis, Talagrand and Haydon have considered aspects of this question.

But no general characterization, in terms of linear topological structure, has emerged.

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Definition and motivation for $(*)$

Definition

A topological space X has $(*)$ if there are families $\mathscr{U}_n$, $n \in \mathbb{N}$, of open sets, such that for any $x, y \in X$, there is $n \in \mathbb{N}$ satisfying

1. $\{x, y\} \cap \bigcup \mathscr{U}_n$ is non-empty
2. $\{x, y\} \cap U$ is at most a singleton for all $U \in \mathscr{U}_n$.

Example

$X = \mathbb{R}$, $\mathscr{U}_n = \{$open intervals of length $n^{-1}\}$.

Spaces having G_δ-diagonals have $(*)$.

There are many compact non-metrizable spaces having $(*)$.

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If a dual Banach space X^* admits a strictly convex *dual* norm, then (B_{X^*}, w^*) has $(*)$.

Theorem

The space X^* admits a strictly convex dual norm if and only if (B_{X^*}, w^*) has $(*)$ *with slices*.

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To what extent can we do without the geometry, i.e. without slices?

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Consequences of $(*)$

We study $C(K)$ spaces because they form a universal class: $X \hookrightarrow C(B_{X^*}, w^*)$.

Theorem

If K is compact and scattered then $C(K)^*$ admits a strictly convex dual norm if and only if K has $(*)$.

Theorem

- If X has $(*)$ then X is fragmentable.
- Countably compact spaces having $(*)$ are compact (generalizes Chaber). In particular, ω_1 does not have $(*)$.
- If L is locally compact and has $(*)$ then $L \cup \{\infty\}$ is countably tight.

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Examples of spaces having $(*)$

Definition (Gruenhage, 1987)

A topological space X, $\mathrm{card}(X) \leq \mathfrak{c}$, is *Gruenhage* if there are open sets U_n, $n \in \mathbb{N}$, such that if $x, y \in X$ then $\{x, y\} \cap U_n$ is a singleton for some n.

Metrizable, σ-discrete and descriptive compact spaces are Gruenhage. The tree $\sigma\mathbb{Q}$ is Gruenhage but not descriptive.

All Gruenhage spaces have $(*)$.

Theorem (Smith, 2009)

If K is Gruenhage compact then $C(K)^*$ admits a strictly convex dual norm.

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- Ostaszewski's space O is scattered but does not have $(*)$, thus $C(O)^*$ does not admit a strictly convex dual norm.
- Using CH or $\mathfrak{b} = \aleph_1$, there exist compact non-Gruenhage spaces having $(*)$, and with cardinality $\aleph_1$.

Proposition

(MA) If L, $\mathrm{card}(L) < \mathfrak{c}$, is locally compact, locally countable and has $(*)$ then L is σ-discrete (and thus Gruenhage).

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ZFC Example

There exists a locally compact non-Gruenhage space D having a G_δ-diagonal.

D is a 'duplicate' $\Lambda \times \{-1, 1\}$, where Λ is Kurepa's tree of injective functions

$$t : \alpha \longrightarrow \mathbb{N}$$

with ordinal domain and co-infinite range.

A Baire category argument can be used to show that D is non-Gruenhage.

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- If K has $(*)$ then does $C(K)^*$ admit a strictly convex dual norm?
- In particular, what if $K = L \cup \{\infty\}$, where L has a G_δ-diagonal?
- If (B_{X^*}, w^*) has $(*)$, does X^* admit a strictly convex dual norm? What if X is also Asplund?
- Is $(*)$ preserved by continuous images of compact spaces, or more generally proper maps?

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- J. Orihuela, R. J. Smith and S. Troyanski, *Strictly convex norms and topology*, Proc. London Math. Soc (forthcoming).
- R. J. Smith, *Strictly convex norms, G_δ-diagonals and non-Gruenhage spaces*, Proc. Amer. Math. Soc. (forthcoming).